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Some properties of the spectral flow in semiriemannian geometry

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Abstract

Let $f(z) = \int_0^1 g(z)[\dot{z}, \dot{z}]\, \mathrm{d}s$ be the action integral on a semiriemannian manifold $(\mathcal{M}, g)$ defined on the space of the curves $z : [0, 1] \to \mathcal{M}$ joining two given points z_0 and z_1. The critical points of f are the geodesics joining z_0 and z_1. Let $s \in [0, 1]$. We study the behavior, in dependence of s, of the eigenvalues of the Hessian form of f evaluated at z, restricted to the interval $[0, s]$. A formula for the derivative of the eigenvalues is given and some applications are shown. © 1998 Elsevier Science B.V.

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1. Introduction and statement of the results

Let $(\mathcal{M}, g)$ be a semiriemannian manifold and consider the following eigenvalue problem:

$$\nabla_s^2 \zeta_\sigma + R(\dot{z}, \zeta_\sigma)\dot{z} = -\lambda(\sigma)\zeta_\sigma, \qquad \zeta_\sigma(0) = 0, \quad \zeta_\sigma(\sigma) = 0, \tag{1}$$

where ∇_s denotes the covariant derivative and R the curvature tensor for the metric g, and z is a geodesic. If $\lambda(\sigma) = 0$, (1) reduces to the Jacobi equations for the geodesic z and $z(\sigma)$ is said to be a conjugate point.

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Clearly, the spectrum $\{\lambda(\sigma)\}$ of the (1) (which in this case reduces to the set of the eigenvalues) is a function of σ. We call this function *spectral flow*.

The first result of this paper concerns the evaluation of the "derivative" of the spectral flow $\{\lambda(\sigma)\}$ for simple eigenvalues. More exactly, we shall prove that, if $\lambda(\sigma)$ is simple, the function $\sigma \mapsto \lambda(\sigma)$ is smooth and

$$\lambda'(\sigma) = -\langle \nabla_s \zeta, \nabla_s \zeta \rangle(\sigma), \tag{2}$$

where ζ is an eigenvector relative to σ.

By formula (2) we can deduce that, in Riemannian geometry, the spectral flow is strictly decreasing . Using this fact, it is possible to give a nice proof of the celebrated Morse index theorem on the number of conjugate points along a geodesic. Also (2) can be used to show that in semiriemannian geometry, the analogous of the Morse index theorem cannot exist in the same form (cf. Remark 3.8).

In this paper we apply (2) to study of the geodesics joining two points in static Lorentzian manifolds of dimension 2. We recall that $(\mathcal{M}, g)$ is a *standard static Lorentzian manifold* if

$$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \qquad g(z)[\zeta, \zeta] = \langle \xi, \xi \rangle_x - \beta(x)\tau^2,$$

where $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$ is a Riemannian manifold, $\beta : \mathcal{M}_0 \to \mathbb{R}$ is a smooth positive scalar field on $\mathcal{M}_0$, $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbb{R}$.

Assume that $\dim \mathcal{M} = 2$, so $\dim \mathcal{M}_0 = 1$. Then $\mathcal{M}_0$ is diffeomorphic to $\mathbb{R}$ or to the unit circle S^1. The following results hold.

Theorem 1.1. *Let $(\mathcal{M}, g)$ be a standard static Lorentzian manifold such that:*
(a) $\dim \mathcal{M} = 2$ *and* $\mathcal{M}_0 = \mathbb{R}$;
(b) $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_R)$ *is complete;*
(c) $\forall x \in \mathcal{M}_0,\ 0 < \beta(x) \le N$.

Then:
(1) *every spacelike geodesic does not have conjugate points;*
(2) *if z_0 and z_1 are causally related, there are no spacelike geodesics joining z_0 and z_1;*
(3) *if z_0 and z_1 are not causally related, there is one and only one geodesic joining z_0 and z_1, and it is spacelike;*
(4) *given any two points and z_0 and z_1 there is at most one spacelike geodesic joining them.*

In particular, by the above theorem, it happens that the points z_0 and z_1 can be joined by either spacelike or causal geodesics.

If $\mathcal{M}_0 = S^1$, by the results of [2], for any couple of points there exist infinitely many spacelike geodesics joining them. On the other hand, they have no conjugate points.

Remark 1.1. Some results on the structure of conjugate points for spacelike geodesics have been obtained in [7]. In particular, it is shown that conjugate points on a spacelike geodesics are unstable.

2. Basic definitions and preliminary results

Let $(\mathcal{M}, g)$ be a semiriemannian manifold, i.e. a smooth finite-dimensional manifold, equipped with a metric tensor g having an index ν, $0 \leq \nu \leq \dim \mathcal{M}$. This means that for any $z \in \mathcal{M}$, $g(z)[\cdot, \cdot]$ is a nondegenerate bilinear form on the tangent space $T_z\mathcal{M}$ at z to $\mathcal{M}$, having exactly ν negative eigenvalues. If $\nu = 0$, $(\mathcal{M}, g)$ is called *Riemannian manifold*, while if $\nu = 1$, $(\mathcal{M}, g)$ is called *Lorentzian manifold*. For more details in semiriemannian geometry see [1,10].

For any interval $[a, b]$ of $\mathbb{R}$ and for any $n \in \mathbb{N}$, we denote by $H^{1,2}([a, b], \mathbb{R}^n)$ the Sobolev space of the absolutely continuous curves, having a square integrable derivative.

Let $\mathcal{M}$ be a smooth manifold, we consider the set $H^{1,2}([0, 1], \mathcal{M})$ of the continuous curves $z : [0, 1] \to \mathcal{M}$ such that for any local chart (U, φ) of the manifold, with $U \cap z([0, 1]) \neq \emptyset$, the curve $z \circ \varphi^{-1} \in H^{1,2}(\varphi(U), \mathbb{R}^n)$, $n = \dim \mathcal{M}$. It is well known (see for instance [11]) that $H^{1,2}([0, 1], \mathcal{M})$ is equipped with a structure of infinte-dimensional Hilbert manifold, modeled on $H^{1,2}([0, 1], \mathbb{R}^n)$. For any $z \in H^{1,2}([0, 1], \mathcal{M})$ the tangent space $T_zH^{1,2}([0, 1], \mathcal{M})$ to $H^{1,2}([0, 1], \mathcal{M})$ at z is identified with

$$T_zH^{1,2}([0, 1], \mathcal{M}) \equiv \{\zeta \in H^{1,2}([0, 1], T\mathcal{M}): \pi \circ \zeta = z\},$$

where $T\mathcal{M}$ is the tangent bundle of $\mathcal{M}$ and $\pi : T\mathcal{M} \to \mathcal{M}$ the projection map.

Now, fix two points z_0 and z_1 in $\mathcal{M}$ and consider the set

$$\Omega^1 \equiv \Omega^1(z_0, z_1, \mathcal{M}) = \{z \in H^{1,2}([0, 1], \mathcal{M}): z(0) = z_0, z(1) = z_1\}. \tag{3}$$

It is not difficult to see that Ω^1 is an infinite-dimensional submanifold of $H^{1,2}([0, 1], \mathcal{M})$ and for any $z \in \Omega^1$ the tangent space $T_z\Omega^1$ is given by

$$T_z\Omega^1 = \{\zeta \in T_zH^{1,2}([0, 1], \mathcal{M}): \zeta(0) = 0, \zeta(1) = 0\}. \tag{4}$$

We consider the *action integral* $f : \Omega^1 \to \mathbb{R}$ defined as

$$f(z) = \int_0^1 g(z(s))[\dot{z}(s), \dot{z}(s)]\, \mathrm{d}s,$$

for any $z \in \Omega^1$. It is well known that f is smooth and its critical points are *geodesics*, i.e. smooth curves satisfying

$$\nabla_s \dot{z} = 0, \tag{5}$$

where $\nabla_s \dot{z}$ is the covariant derivative of $\dot{z}$ along z (induced by the metric structure g, cf. [10]).

By (5) one immediately deduces the existence of a real constant E_z such that for any $z \in [0, 1]$,

$$E_z = g(z(s))[\dot{z}(s), \dot{z}(s)]. \tag{6}$$

Then the geodesic z is called *spacelike, lightlike* or *timelike*, if E_z is positive, null or negative, respectively.

Since z is a critical point of f it is the well-defined Hessian form of f at z,

$$f''(z)[\zeta,\zeta] = \frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}(f(\eta(\lambda,\cdot)))_{|\lambda=0},$$

where $\eta(\lambda, s)$ is a two-parameter map such that $\eta(0,s) = z(s)$, $\eta_\lambda(0,s) = \zeta(s)$, and $\zeta \in T_z\Omega^1$.

A standard computation shows that

$$f''(z)[\zeta,\zeta] = \int_0^1 (\langle \nabla_s\zeta, \nabla_s\zeta\rangle - \langle R(\dot z,\zeta)\dot z,\zeta\rangle)\,\mathrm{d}s, \tag{7}$$

where R denotes the curvature tensor for the metric and we have denoted the metric g by $\langle\cdot,\cdot\rangle$.

Integrating by parts in (7) shows that a vector field $\zeta \neq 0$ is in the kernel of $f''(z)$ if and only if

$$\nabla_s^2\zeta + R(\dot z,\zeta)\dot z = 0, \quad \zeta(0) = 0, \quad \zeta(1) = 0. \tag{8}$$

Then ζ is in the kernel of $f''(z)$ if and only if ζ is a *Jacobi field* (cf. [10]) with null boundary conditions.

Let z be a critical point of f, for any $s \in]0,1]$ consider the functional f restricted to the interval $[0, s]$:

$$f_s(w) = \int_0^s \langle \dot w, \dot w\rangle\,\mathrm{d}s,$$

defined on the manifold

$$\Omega_s^1(z) = \{w \in H^{1,2}([0,s],\mathcal{M})\colon w(0) = z_0,\ w(s) = z(s)\}.$$

Note that $z_s = z_{|[0,s]}$ is a critical point of f_s on $\Omega_s^1(z)$. We recall that a point $z(s)$, $s \in]0,1]$, is said to *conjugate* to z_0 along z if the kernel of $f_s''(z_s)$ is nontrivial and the dimension of this kernel is called *multiplicity* to the conjugate point $z(s)$ (by (8) the multiplicity of a conjugate point is always finite). The index $\mu(z)$ of the geodesic z is the number of conjugate points $z(s)$, $s \in]0,1[$, to z_0 along z, counted with their multiplicity.

A very famous theorem due to M. Morse shows that in the Riemannian case, the Morse index of z (i.e. the maximal dimension of a subspace where $f''(z)$ is negative definite) is equal to the index $\mu(z)$. These results can be proved studying the behavior of the eigenvalues $\lambda(\sigma)$ of the problem

$$\nabla_s^2\zeta_\sigma + R(\dot z,\zeta_\sigma)\dot z = -\lambda(\sigma)\zeta_\sigma, \qquad \zeta_\sigma(0) = 0, \quad \zeta_\sigma(\sigma) = 0, \tag{9}$$

$\sigma \in]0,1]$, whenever $\lambda(\sigma)$ is nearby 0.

In this paper we give a formula for the derivative of $\lambda(\sigma)$ in problem (9) for any semiriemannian manifold. Such a formula shows that, in the non-Riemannian case, the above result in general does not hold.

Using the derivative of $\lambda(\sigma)$ we shall deduce some results for Lorentzian manifolds, in particular for standard static ones.

3. The main theorem and first consequences

In this section we fix a semiriemannian manifold $(\mathcal{M}, \langle\cdot,\cdot\rangle_z)$, two points z_0, z_1 of $\mathcal{M}$ and a geodesic $z : [0,1] \to \mathcal{M}$ joining z_0 and z_1. Let us consider the following problem which is the weak formulation of problem (9)

$$\begin{aligned} &\int_0^\sigma (\langle\nabla_s\zeta, \nabla_s\zeta'\rangle - \langle R(\dot z,\zeta)\dot z, \zeta'\rangle)\,\mathrm{d}s = \lambda(\sigma)\int_0^\sigma \langle\zeta,\zeta'\rangle_R\,\mathrm{d}s \\ &\zeta(0) = 0, \quad \zeta(\sigma) = 0, \end{aligned} \tag{10}$$

for any $\zeta' \in T_{z_\sigma}\Omega^1_s(z)$, where $\langle\cdot,\cdot\rangle_R$ is an auxiliary Riemannian metric on $\mathcal{M}$, $\zeta \in T_{z_\sigma}\Omega^1_s(z)$. In this section we get a formula for the first derivative of $\lambda(\sigma)$ with respect to σ, giving some simple consequences of it. To avoid technical difficulties we shall only consider the case of simple eigenvalues. For multiple eigenvalues analogous results can be obtained studying

$$\liminf_{\delta\to 0}\frac{\lambda(\sigma+\delta)-\lambda(\sigma)}{\delta} \quad \text{and} \quad \limsup_{\delta\to 0}\frac{\lambda(\sigma+\delta)-\lambda(\sigma)}{\delta}.$$

We need the following simple lemma.

Lemma 3.1. *Let H be a Hilbert space, $\mathcal{L}(H)$ the space of the continuous linear operators from H into itself and let $A : [0,1] \to \mathcal{L}(H)$ be a smooth map. Let $\sigma_0 \in]0,1[$, λ_0 a simple eigenvalue of $A(\sigma_0)$ and $v_0 \in H$ such that $A(\sigma_0)\, v_0 = \lambda_0 v_0$ and $\|v_0\| = 1$, where $\|\cdot\|$ is the norm of H.*

Then there exists a neighborhood U of σ and smooth maps $\lambda(\sigma)$ and $v(\sigma)$ defined on U, such that for any $\sigma \in U$, $\lambda(\sigma)$ is a simple eigenvalue of $A(\sigma)$, $\|v(\sigma)\| = 1$, $\lambda(\sigma_0) = \lambda_0$ and $v(\sigma_0) = v_0$.

Proof. Let $F : \mathbb{R} \times H \times]0,1[\to \mathbb{R} \times H$ be the map such that

$$F(\lambda, v, \sigma) = (A(\sigma)v - \lambda v, \|v\|^2 - 1).$$

Since the partial derivative $(\partial F/\partial(\lambda, v))(\lambda_0, v_0, \sigma_0) : \mathbb{R}\times H \to \mathbb{R}\times H$ is a linear isomorphism, the Implicit function theorem gives the proof. □

Using Lemma 3.1 we obtain the following result for problem (10).

Theorem 3.2. *Let $\sigma_0 \in]0,1[$, $\lambda_0 = \lambda(\sigma_0)$ be a simple eigenvalue of (10) and $\zeta_0 = \zeta(s,\sigma_0)$ be a normalized associated eigenfunction, i.e. $\int_0^{\sigma_0}\langle\zeta_0,\zeta_0\rangle_R\,\mathrm{d}s = 1$.*

Then there exists a neighborhood U *of* σ_0 *and two smooth functions* $\lambda(\sigma) : U \to \mathbb{R}$, $\zeta_\sigma = \zeta(s, \sigma) : U \to T\mathcal{M}$ *such that:*
– *for any* $\sigma \in U$, $\lambda(\sigma)$ *is a simple eigenvalue of* (10),
– $\zeta_\sigma \in T_{z(\sigma)}\Omega^1_s(z)$ *is a normalized associated eigenfunction, and*

$$\lambda'(\sigma_0) = -\langle \nabla_s \zeta_0(\sigma_0), \nabla_s \zeta_0(\sigma_0)\rangle_z. \tag{11}$$

Proof. Let $\lambda(\sigma)$ and $\zeta_\sigma = \zeta(s, \sigma)$ as in Lemma 3.1. Since $\int_0^\sigma \langle \zeta, \zeta\rangle_R \, \mathrm{d}s = 1$, by (10) we have

$$\lambda(\sigma) = \int_0^\sigma (\langle \nabla_s \zeta, \nabla_s \zeta'\rangle_z - \langle R(\dot z, \zeta)\dot z, \zeta'\rangle_z)\, \mathrm{d}s.$$

Let $A(z) : T_z\mathcal{M} \to T_z\mathcal{M}$ be the linear operator such that

$$\langle A(z)v, v'\rangle_z = \langle v, v'\rangle_R \quad \forall v, v' \in T_z\mathcal{M}.$$

Integrating by parts in (10) gives

$$\begin{aligned} &\nabla_s^2 \zeta_\sigma + R(\dot z, \zeta_\sigma)\dot z = -\lambda(\sigma) A(z(s))\zeta_\sigma, \\ &\zeta_\sigma(0) = 0, \qquad \zeta_\sigma(\sigma) = 0. \end{aligned} \tag{12}$$

Differentiating gives

$$\begin{aligned} \lambda'(\sigma) &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left(\int_0^\sigma (\langle \nabla_s \zeta, \nabla_s \zeta\rangle_z - \langle R(\dot z, \zeta)\dot z, \zeta\rangle_z)\, \mathrm{d}s\right) \\ &= \langle \nabla_s \zeta(\sigma, \sigma), \nabla_s \zeta(\sigma, \sigma)\rangle_z - \langle R(\dot z(\sigma), \zeta(\sigma, \sigma))\dot z(\sigma), \zeta(\sigma, \sigma)\rangle_z \\ &\quad + \int_0^\sigma 2\langle \nabla_\sigma \nabla_s \zeta, \nabla_s \zeta\rangle_z \, \mathrm{d}s - \int_0^\sigma \frac{\mathrm{d}}{\mathrm{d}\sigma}(\langle R(\dot z, \zeta)\dot z, \zeta\rangle_z)\, \mathrm{d}s. \end{aligned} \tag{13}$$

We claim that

$$\nabla_\sigma \nabla_s \zeta = \nabla_s \nabla_\sigma \zeta. \tag{14}$$

To prove (14), consider the three-parameter map $\Gamma(s, \sigma, \mu)$ such that

$$\Gamma(s, \sigma, 0) = z(s), \qquad \frac{\partial \Gamma}{\partial \mu}(s, \sigma, 0) = \zeta(s, \sigma).$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\sigma}\langle \nabla_s \zeta(s, \sigma), \nabla_s \zeta(s, \sigma)\rangle_z &= \frac{\mathrm{d}}{\mathrm{d}\sigma}\left\langle \nabla_s \frac{\partial \Gamma}{\partial \mu}(s, \sigma, 0), \nabla_s \frac{\partial \Gamma}{\partial \mu}(s, \sigma, 0)\right\rangle_z \\ &= 2\left\langle \nabla_\sigma \nabla_s \frac{\partial \Gamma}{\partial \mu}(s, \sigma, 0), \nabla_s \frac{\partial \Gamma}{\partial \mu}(s, \sigma, 0)\right\rangle_z. \end{aligned}$$

Now, by a well-known formula in semiriemannian geometry (cf. [10]),

$$\nabla_\sigma \nabla_s \frac{\partial \Gamma}{\partial \mu}(s,\sigma,0) = \nabla_s \nabla_\sigma \frac{\partial \Gamma}{\partial \mu}(s,\sigma,0) + R\left(\frac{\partial \Gamma}{\partial s}(s,\sigma,0), \frac{\partial \Gamma}{\partial \sigma}(s,\sigma,0)\right)\frac{\partial \Gamma}{\partial \mu}(s,\sigma,0).$$

On the other hand $(\partial \Gamma/\partial\sigma)(s,\sigma,0) = 0$, since Γ does not depend on σ. Then (14) holds.

Now, $\langle R(\zeta(\sigma,\sigma),\dot z(\sigma))\dot z(\sigma),\zeta(\sigma,\sigma)\rangle_z = 0$ (since $\zeta(\sigma,\sigma) = 0$). Then by (13) and the symmetric properties of the tensor R,

$$\lambda'(\sigma) = \langle \nabla_s\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z + \int_0^\sigma (2\langle \nabla_s\nabla_\sigma\zeta, \nabla_s\zeta\rangle_z - 2\langle R(\dot z, \nabla_\sigma\zeta)\dot z, \zeta\rangle_z)\,\mathrm{d}s.$$

Integrating by parts and recalling the symmetric properties of the tensor R gives

$$\begin{aligned}\lambda'(\sigma) = {} & \langle \nabla_s\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z \\ & + 2\langle \nabla_\sigma\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z - 2\langle \nabla_\sigma\zeta(0,\sigma), \nabla_s\zeta(0,\sigma)\rangle_z \\ & - \int_0^\sigma (2\langle \nabla_\sigma\zeta, \nabla_s^2\zeta\rangle_z + 2\langle R(\dot z,\zeta)\dot z, \nabla_\sigma\zeta\rangle_z)\,\mathrm{d}s.\end{aligned}$$

Then, from (10),

$$\begin{aligned}\lambda'(\sigma) = {} & \langle \nabla_s\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z \\ & + 2\langle \nabla_\sigma\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z - 2\langle \nabla_\sigma\zeta(0,\sigma), \nabla_s\zeta(0,\sigma)\rangle_z \\ & + 2\int_0^\sigma \langle \nabla_\sigma\zeta, \lambda(\sigma)A(z(s))\zeta(s,\sigma)\rangle_z\,\mathrm{d}s.\end{aligned}$$

Now, the boundary conditions in (9) give $\zeta(r,r) = 0$ for any r and differentiating gives

$$\nabla_\sigma\zeta(r,r) + \nabla_s\zeta(r,r) = 0,$$

hence

$$\nabla_\sigma\zeta(r,r) = -\nabla_s\zeta(r,r).$$

Moreover, since $\zeta(0,\sigma) = 0$ for any σ, differentiating gives

$$\nabla_\sigma\zeta(0,\sigma) = 0.$$

Therefore

$$\lambda'(\sigma) = -\langle \nabla_s\zeta(\sigma,\sigma), \nabla_s\zeta(\sigma,\sigma)\rangle_z + 2\lambda(\sigma)\int_0^\sigma \langle \nabla_\sigma\zeta, A(z(s))\zeta(s,\sigma)\rangle_z\,\mathrm{d}s.$$

Now, since $\zeta(\cdot,\sigma)$ is normalized,

$$1=\int_0^\sigma \langle\zeta,\zeta\rangle_R\,ds=\int_0^\sigma \langle A(z)\zeta,\zeta\rangle_z\,ds.$$

Differentiating with respect to σ, since $A(z)$ is symmetric and $\zeta(\sigma,\sigma)=0$, gives

$$\begin{aligned}0&=\langle A(z(\sigma))\zeta(\sigma,\sigma)\zeta(\sigma,\sigma)\rangle_z+2\int_0^\sigma \langle A(z)\zeta,\nabla_\sigma\zeta\rangle_z\,ds\\ &=2\int_0^\sigma \langle A(z)\zeta,\nabla_\sigma\zeta\rangle_z\,ds,\end{aligned}$$

and the proof is complete. □

The following result is well-known. We report the simple proof for the sake of completeness.

Lemma 3.3. *Let ζ be a Jacobi field along z satisfying* (2.7). *Then*

$$\langle\dot z,\nabla_s\zeta\rangle_z=\langle\dot z,\zeta\rangle_z=0\quad\forall s\in[0,\sigma].$$

Proof. Take $\varphi(s)=\langle\dot z(s),\zeta(s)\rangle_z$. Differentiating gives

$$\begin{aligned}\varphi'(s)&=\langle\dot z(s),\nabla_s\zeta(s)\rangle_z\\ \varphi''(s)&=\langle\dot z(s),\nabla_s^2\zeta(s)\rangle_z=-\langle\dot z(s),R(\dot z(s),\zeta(s))\dot z(s)\rangle_z=0,\end{aligned}$$

because of the antisymmetry of the curvature tensor. Since $\varphi(0)=\varphi(\sigma)=0$ and $\varphi''\equiv 0$, we deduce that $\varphi\equiv 0$ and $\varphi'\equiv 0$. □

Remark 3.4. If the eigenvalues of $f''(z)$ are not simple, it is possible to see that the set of the limit points of the Newton quotient is given by

$$\{-\langle\nabla_s\zeta(\sigma,\sigma),\nabla_s\zeta(\sigma,\sigma)\rangle_z,\ \zeta \text{ satisfies (3.1)}\}.$$

An immediate consequence of Theorem 3.2 is a different proof that the eigenvalues are strictly decreasing in the Riemannian case (see [8]).

Another simple consequence of Theorem 3.2 is the following:

Theorem 3.5. *Assume that $(\mathcal{M},\langle\cdot,\cdot\rangle_z)$ is Lorentzian and z be timelike (i.e. $\langle\dot z,\dot z\rangle_z<0$ for any s). Then the eigenvalues are strictly decreasing.*

Proof. Let ζ be a Jacobi field satisfying (1.1), by Lemma 3.3, $\langle\dot z,\nabla_s\zeta\rangle_z=0$, so $\nabla_s\zeta$ is orthogonal to $\dot z$. Since $\dot z$ is timelike, $\nabla_s\zeta$ is spacelike, i.e. $\langle\nabla_s\zeta,\nabla_s\zeta\rangle_z>0$, so $\lambda'(\sigma)<0$ for any σ. □

Remark 3.6. Suppose now that is lightlike (i.e. $\langle \dot{z}, \dot{z} \rangle_z \equiv 0$). Approximating z by timelike geodesics, it is possible to prove that the eigenvalues $\lambda(\sigma)$ are nonincreasing and there are no intervals contained in $[0, 1]$ where $\lambda(\sigma)$ is constant and null. Indeed if $z(\sigma_0)$ is conjugate to $z(0)$ along z (i.e. $\lambda(\sigma_0) = 0$), it is isolated (cf. e.g. [10, p. 299]).

Remark 3.7. If $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ is a two-dimensional Lorentzian manifold and z is a spacelike geodesic (i.e. $\langle \dot{z}, \dot{z} \rangle_z > 0$), then $\lambda(\sigma)$ is strictly increasing. Indeed in this case $\nabla_s \zeta$ is orthogonal to a spacelike vector, so (since $\dim \mathcal{M} = 2$) $\nabla_s \zeta$ is timelike. Then $\lambda'(\sigma) = -\langle \nabla_s \zeta, \nabla_s \zeta \rangle_z > 0$.

Remark 3.8. By Theorem 3.5 it is possible to get an index theorem considering only the timelike or lightlike geodesic. It is just the same as in the Riemannian case. Moreover Theorem 3.2 shows that, in general, for spacelike geodesics the situation is more complicated.

4. The static case

In this section we shall apply the results of Section 3 to a standard static Lorentzian manifold. We recall that a (standard) static Lorentzian manifold $(\mathcal{M}, \langle \cdot, \cdot \rangle_z)$ satisfies

$$\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}, \qquad \langle \zeta, \zeta \rangle_z = \langle \xi, \xi \rangle_x - \beta(x)\tau^2,$$

where $(\mathcal{M}_0, \langle \cdot, \cdot \rangle_x)$ is a Riemannian manifold, $\beta : \mathcal{M}_0 \to \mathbb{R}$ is a smooth positive scalar field on $\mathcal{M}_0$, $z = (x, t) \in \mathcal{M}$ and $\zeta = (\xi, \tau) \in T_z\mathcal{M} = T_x\mathcal{M}_0 \times \mathbb{R}$.

The following lemma is needed to "translate" Theorem 3.2 in the coordinates (x, t).

Lemma 4.1. *Let z be a geodesic in $\mathcal{M}$ and $\zeta(s) = (\xi(s), \tau(s))$ a Jacobi field satisfying* (1.1). *Then*

$$\langle D_s\zeta, D_s\zeta \rangle_z(\sigma) = \langle \nabla_s\xi, \nabla_s\xi \rangle_x(\sigma) - \beta(x(\sigma))\dot{\tau}^2(\sigma), \tag{15}$$

where D_s is the covariant (Lorentzian) derivative with respect to $\langle \cdot, \cdot \rangle_z$ and ∇_s is the covariant (Riemannian) derivative with respect to $\langle \cdot, \cdot \rangle_x$.

Proof. Let $\varphi(s) = \langle \zeta(s), \zeta(s) \rangle_z$, since ζ is a Jacobi field, differentiating gives:

$$\begin{aligned} \varphi'(s) &= 2\langle \zeta(s), \nabla_s\zeta(s) \rangle_z, \\ \varphi''(s) &= 2\langle D_s\zeta(s), D_s\zeta(s) \rangle_z + 2\langle \zeta(s), \nabla_s^2\zeta(s) \rangle_z, \\ &= 2\langle D_s\zeta(s), D_s\zeta(s) \rangle_z - 2\langle \zeta(s), R(\dot{z}(s), \zeta(s))\dot{z}(s) \rangle_z. \end{aligned}$$

Evaluating at $s = \sigma$, since $\zeta(\sigma) = 0$, gives

$$\langle D_s\zeta, D_s\zeta \rangle_z(\sigma) = \left[\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}s^2} \varphi(s) \right]_{s=\sigma}. \tag{16}$$

On the other hand, computing in static coordinates gives $\varphi(s) = \langle\zeta(s), \zeta(s)\rangle_z = \langle\xi, \xi\rangle_x - \beta(x)\tau^2$. Differentiating gives

$$\varphi'(s) = 2\langle\xi, \nabla_s\xi\rangle_x - \langle\nabla\beta(x), \dot{x}\rangle_x\tau^2 - 2\beta(x)\tau\dot{\tau},$$

where $\nabla\beta(x)$ is the gradient of β with respect to the Riemannian structure on $\mathcal{M}_0$. Differentiating again gives

$$\begin{aligned}\varphi''(s) = {} & 2\langle\nabla_s\xi, \nabla_s\xi\rangle_x + 2\langle\xi, \nabla_s^2\xi\rangle_x - 4\langle\nabla\beta(x), \dot{x}\rangle_x\tau\dot{\tau} \\ & - H_\beta(x)[\xi, \dot{x}]\tau - \langle\nabla\beta(x), \nabla_s\dot{x}\rangle_x\tau^2 - 2\beta(x)\dot{\tau}^2 - 2\beta(x)\tau\ddot{\tau},\end{aligned}$$

where $H_\beta(x)$ is the Hessian of β with respect to the Riemannian structure of $\mathcal{M}_0$. Then, evaluating at $s = \sigma$, since $\xi(\sigma) = 0$ and $\tau(\sigma) = 0$, and using (16) gives the conclusion. □

Remark 4.2. The analogous of (15)

$$\langle D_s\zeta, D_s\zeta\rangle_z(\sigma) = \langle\alpha(x(\sigma), t(\sigma))\nabla_s\xi, \nabla_s\xi\rangle_x(\sigma) - \beta(x(\sigma), t(\sigma))\dot{\tau}^2(\sigma) \tag{17}$$

is still valid for a Lorentzian manifolds which admit an orthogonal splitting.

We recall that an orthogonal splitting is a Lorentzian manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ with the metric tensor

$$\langle\zeta, \zeta\rangle_z = \langle\alpha(x, t)\xi, \xi\rangle_x - \beta((x, t))\tau^2,$$

where $\alpha(x, t)$ is a positive linear operator on $T_x\mathcal{M}_0$, smoothly depending on $z = (x, t) \in \mathcal{M}$.

Another case in which the general formula (2) takes a simple form is in the *standard stationary* case. We recall that a product manifold $\mathcal{M} = \mathcal{M}_0 \times \mathbb{R}$ is called a standard stationary if the metric tensor has the form:

$$\langle\zeta, \zeta\rangle_z = \langle\xi, \xi\rangle_x + 2\langle\delta(x), \xi\rangle_x\tau - \beta(x)\tau^2,$$

where $\delta(x)$ is a smooth vector field on $\mathcal{M}_0$. In this case we have

$$\begin{aligned}\langle D_s\zeta, D_s\zeta\rangle_z(\sigma) = {} & \langle\nabla_s\xi, \nabla_s\xi\rangle_x(\sigma) + 2\langle\delta(x), \nabla_s\xi\rangle_x(\sigma)\tau(\sigma) \\ & - \beta(x(\sigma))\dot{\tau}^2(\sigma).\end{aligned} \tag{18}$$

In order to prove (17) and (18), it is sufficient to carry out the relative computations. Since they are simple, we omit them.

We fix now two points $z_0 = (x_0, t_0)$, $z_1 = (x_1, t_1)$ in the static Lorentzian manifold $(\mathcal{M}, \langle\cdot, \cdot\rangle_z)$, the geodesics joining z_0 and z_1 are the critical points of the action integral

$$f(z) = \frac{1}{2}\int_0^1 \langle\dot{z}, \dot{z}\rangle_z \, ds = \frac{1}{2}\int_0^1 [\langle\dot{x}, \dot{x}\rangle_x - \beta(x)\dot{t}^2] \, ds$$

on the manifold

$$\mathcal{Z}^1 = \Omega^1(z_0, z_1, \mathcal{M}) = \Omega^1(x_0, x_1, \mathcal{M}_0) \times \Omega^1(t_0, t_1, \mathbb{R}).$$

The search of the critical points of the action integral can be reduced (in the static case) to the search of the critical points of the functional $J : \Omega^1(x_0, x_1, \mathcal{M}_0) \to \mathbb{R}$ defined by (cf. [2])

$$J(x) = \int_0^1 \langle \dot{x}, \dot{x} \rangle_x \, ds - \frac{(t_1 - t_0)^2}{\int_0^1 (1/\beta(x)) \, ds} \tag{19}$$

(an analogous result holds for stationary Lorentzian manifolds, see [5]). Indeed a curve $z = (x, t) \in \Omega^1(z_0, z_1, \mathcal{M})$ is a critical point of f if and only if x is a critical point of J on $\Omega^1(x_0, x_1, \mathcal{M}_0)$ and t is the unique solution of the problem

$$\frac{d}{ds}(\beta(x)\dot{t}) = 0, \qquad t(0) = t_0, \quad t(1) = t_1. \tag{20}$$

Solving (20) gives

$$t(s) = t_0 + \frac{t_1 - t_0}{\int_0^1 (1/\beta(x)) \, ds} \int_0^r \frac{1}{\beta(x)} \, dr. \tag{21}$$

We set

$$k_x = \frac{t_1 - t_0}{\int_0^1 (1/\beta(x)) \, ds}. \tag{22}$$

Now, for any $\sigma \in]0, 1[$ let $z_\sigma = (x_\sigma, t_\sigma)$ be the restriction of z to $[0, \sigma]$. The curve z_σ is a critical point of the functional

$$f_\sigma(y, \theta) = \int_0^\sigma (\langle \dot{y}, \dot{y} \rangle_R - \beta(y)\dot{\theta}^2) \, ds, \tag{23}$$

on the manifold

$$\mathcal{Z}_\sigma^1 = \Omega_\sigma^1(z_0, z(\sigma), \mathcal{M}) = \Omega_\sigma^1(x_0, x(\sigma), \mathcal{M}_0) \times \Omega_\sigma^1(t_0, t(\sigma), \mathbb{R}),$$

where

$$\Omega_\sigma^1(x_0, x(\sigma), \mathcal{M}_0) = \{y \in H^{1,2}([0, \sigma], \mathcal{M}_0) \colon y(0) = x_0, \, y(\sigma) = x(\sigma)\}$$

and

$$\Omega_\sigma^1(t_0, t(\sigma), \mathbb{R}) = \{\theta \in H^{1,2}([0, \sigma], \mathbb{R}) \colon \theta(0) = t_0, \, \theta(\sigma) = t(\sigma)\}.$$

A similar variational principle holds in $\mathcal{Z}_\sigma^1$ (cf. [3]), considering the functional $J_\sigma : \Omega_\sigma^1(x_0, x(\sigma), \mathcal{M}_0) \to \mathbb{R}$ given by

$$J_\sigma(y) = \int_0^\sigma \langle \dot{y}, \dot{y} \rangle_R \, ds - \frac{(t(\sigma) - t_0)^2}{\int_0^\sigma (1/\beta(y)) \, ds}. \tag{24}$$

Indeed, z_σ is a critical point of f_σ if and only if x_σ is a critical point of J_σ and t_σ solves

$$\frac{\mathrm{d}}{\mathrm{d}s}(\beta(x_\sigma)\dot{t}_\sigma) = 0, \qquad t_\sigma(0) = t_0, \quad t_\sigma(\sigma) = t(\sigma),$$

i.e.

$$t_\sigma(s) = t_0 + \frac{t(\sigma) - t_0}{\int_0^\sigma (1/\beta(x_\sigma))\,\mathrm{d}s} \int_0^s \frac{1}{\beta(x_\sigma)}\,\mathrm{d}r \equiv \phi_\sigma(x). \tag{25}$$

Note that $\phi_\sigma : \Omega^1_\sigma(x_0, x(\sigma), \mathcal{M}_0) \to \Omega^1_\sigma(t_0, t(\sigma), \mathbb{R})$ and

$$\frac{t(\sigma) - t_0}{\int_0^\sigma (1/\beta(x_\sigma))\,\mathrm{d}s} = k_x \quad \forall \sigma \in]0, 1]. \tag{26}$$

A nonnull vector field ξ along x_σ (with $\xi(0) = 0$ and $\xi(\sigma) = 0$) is said to be a *Jacobi field along* x_σ if

$$J''_\sigma(x_\sigma)[\xi, \xi'] = 0 \quad \forall \xi' \in T_{x_\sigma}\Omega^1_\sigma(x_0, x(\sigma), \mathcal{M}_0). \tag{27}$$

In this case we say that $x(\sigma)$ is *conjugate to* x_0 along x. The dimension of ker $J''_\sigma(x_\sigma)$ is called the *multiplicity* of the conjugate point. As proved in [3], a vector field $\zeta = (\xi, \tau)$ is a Jacobi field along z_σ with $\zeta(0) = 0$ and $\zeta(\sigma) = 0$, if and only if ξ is a Jacobi field along x_σ and τ is the solution of the Cauchy problem

$$\frac{\mathrm{d}}{\mathrm{d}s}(\langle \nabla\beta(x), \xi\rangle_R \dot{t} + \beta(x)\dot{\tau}) = 0, \qquad \tau(0) = t_0, \quad \tau(\sigma) = 0. \tag{28}$$

Moreover the multiplicity of $x(\sigma)$ is the same as the multiplicity of $z(\sigma)$.

Let $\mu(\sigma)$ be an eigenvalue of $J''_\sigma(x_\sigma)$ and ξ_σ be a normalized eigenvector associated to $\mu(\sigma)$, i.e.

$$J''_\sigma(x_\sigma)[\xi_\sigma, \xi'_\sigma] = \mu(\sigma) \int_0^\sigma \langle \xi_\sigma, \xi'_\sigma\rangle_R\,\mathrm{d}s \quad \forall \xi' \in T_{x_\sigma}\Omega^1_\sigma(x_0, x(\sigma), \mathcal{M}_0), \tag{29}$$

and

$$\int_0^\sigma \langle \xi_\sigma, \xi_\sigma\rangle_R\,\mathrm{d}s = 1.$$

About the behavior of $\mu(\sigma)$, the analogous of Theorem 3.1 holds.

Theorem 4.3. *Let* $\sigma_0 \in]0, 1]$ *and let* $x(\sigma)$ *be a conjugate point along* x *(having multiplicity* 1*). Then*

$$\mu'(\sigma_0) = -\langle \nabla_s\xi_{\sigma_0}(\sigma_0), \nabla_s\xi_{\sigma_0}(\sigma_0)\rangle_R + \beta(x(\sigma_0))\dot{\tau}_{\sigma_0}(\sigma_0)^2,$$

where $\tau_{\sigma_0} = \phi'_{\sigma_0}(x_{\sigma_0})[\xi_{\sigma_0}]$ *(cf.* (25)).

Notice that in Theorem 3.3 we are assuming $\mu(\sigma_0) = 0$.

To get the proof of Theorem 4.3, the following lemma is needed.

Lemma 4.4. *Let* $\xi \in T_{x_\sigma}\Omega^1_\sigma(x_0, x(\sigma), \mathcal{M}_0)$, *then*

$$J''_\sigma(x_\sigma)[\xi, \xi] = f''_\sigma(z_\sigma)[\zeta, \zeta],$$

where $\zeta = (\xi, \tau)$ *and* $\tau = \phi'_\sigma(x_\sigma)[\xi]$.

Proof. Clearly we can reduce to the case $\sigma = 1$. Since $J(x) = f(x, \phi(x))$, differentiating and setting $\tau = \phi'(x)[\xi]$ gives

$$J'(x)[\xi] = f_x(x, \phi(x))[\xi] + f_t(x, \phi(x))[\phi'(x)[\xi]],$$

because $f_t(x, \phi(x)) = 0$. Differentiating again gives

$$J''(x)[\xi, \xi] = f_{xx}(x, \phi(x))[\xi, \xi] + f_{xt}(x, \phi(x))[\xi, \phi'(x)[\xi]].$$

On the other hand,

$$f''(z)\,[(\xi, \tau), (\xi, \tau)] = f_{xx}(z)\,[\xi, \xi] + 2f_{xt}(z)\,[\xi, \tau] + f_{tt}(z)\,[\tau, \tau]\,.$$

Since for any y,

$$f_t(y, \phi(y))\,[\tau] = 0,$$

differentiating with respect to y and evaluating at x gives

$$f_{xt}(z)[\xi, \tau] + f_{xt}(z)[\phi'(x)\xi, \tau] = 0.$$

Since $\tau = \phi'(x)[\xi]$ we obtain the equality. □

Proof of Theorem 4.3. Since $\mu(\sigma) = J''_\sigma(x_\sigma)\,[\xi_\sigma, \xi_\sigma]$, by Lemma 4.4 we get $\mu(\sigma) = f''_\sigma(x_\sigma)[\zeta_\sigma, \zeta_\sigma]$, where $\zeta_\sigma = (\xi_\sigma, \phi'_\sigma(x_\sigma)\,[\xi_\sigma])$. Then the proof follows immediately applying Theorem 3.2 and Lemma 4.1. □

Finally we can prove Theorem 1.1.

Proof of Theorem 1.1.

(1) Standard computations show that if σ is sufficiently small, $J''_\sigma(x_\sigma)$ is positive definite (cf. [9]). Therefore if x has a conjugate point and $x(\bar{\sigma})$ is the "first" conjugate point, we should have

$$\lambda(\sigma) > 0 \ \text{ if } \sigma < \bar{\sigma} \quad \text{and} \quad \lambda(\bar{\sigma}) = 0. \tag{30}$$

But since $\dot{z}$ is spacelike and $\dim \mathcal{M} = 2$, by Lemma 3.3 we have that $D_s\zeta_{\bar{\sigma}}$ is timelike. Then by Theorem 3.2 and Lemma 4.4, $\lambda'(\bar{\sigma}) > 0$, in contradiction with (30).

(2) Under our assumptions, the Lorentzian manifold $(\mathcal{M}, g)$ is globally hyperbolic (cf. [6]), therefore if z_0 and z_1 are causally related (i.e. there exists a timelike curve joining

them), there is a timelike geodesic joining them. Then J has a critical point x such that $J(x) < 0$. Now if by contradiction there is a spacelike geodesic joining z_0 and z_1, there is also a critical point y of J, with $J(y) > 0$. Moreover, by (1) and Lemma 4.4, y is a local minimum point.

Since under our assumptions the functional J satisfies the Palais–Smale condition (cf. [2]), it follows by classical results of calculus of variations (see for instance [8]) that there is a third critical point w of J, such that $J''(w)$ has an eigenvalue negative or null and $J(w) > J(y) > 0$. Then $(w, \phi(w))$ is a spacelike geodesic with conjugate points, in contradiction with (1).

(3) If z_0 and z_1 are not causally related, any geodesic joining z_0 and z_1 is spacelike. Moreover, at least one (spacelike) geodesic joining z_0 and z_1 exists, since J achieves its minimum on $\Omega^1(x_0, x_1, \mathcal{M}_0)$. If by contradiction there are at least two spacelike geodesics, by (1) they are local minimum, so as in (2), there is a third critical point w of J, with $J(w) > 0$ and such that $J''(w)$ has an eigenvalue negative or null, in contradiction with (1).

(4) It follows immediately from (2) and (3).

This completes the proof. □

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